

Qubit flip game on a Heisenberg spin chain

J.A. Miszczak · P. Gawron · Z. Puchała

02/08/2011 (v. 0.72)

Abstract We study a quantum version of a penny flip game played using control parameters of the Hamiltonian in the Heisenberg model. Moreover, we extend this game by introducing auxiliary spins which can be used to alter the behaviour of the system. We show that a player aware of the complex structure of the system used to implement the game can use this knowledge to gain higher mean payoff.

Keywords quantum information; quantum games; quantum control; Heisenberg model; spin chain

1 Introduction

Game theory is used to describe the situations of mutual interaction of several parties, where each party aims to maximize its gain. If one considers the physical system used to define a game, it is reasonable to ask about the influence of physical laws on games. As a result, one should consider more general notion of a game, where strategies are defined in terms of quantum evolution and the system is described in terms of quantum states. In such scheme the payoff function is defined as a mean value of some chosen observables. As players in quantum games can use larger space of strategies this should influence the expected value of the payoff [14, 4].

The first example of a quantum game was provided in the form of a simple penny flip game [14]. In this scenario a player with the ability to use quantum strategies (moves) can always win with a player using only classical moves. Eisert *et al.* [4] provided a more elaborated example of a two-person quantum game by constructing a quantized version of Prisoner's dilemma game. Recent

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developments in this area include the application of quantum games for the analysis of quantum walks [9, 13, 1] and the analysis of quantum games in the presence of decoherence [5, 7, 8] with the special attention to the degradation of entanglement [10, 18, 19]. Quantum games also provide relatively easy scenarios for implementing quantum information processing in physical systems [16] and can be used to probe the influence of decoherence in such systems [6].

The main aim of this work is to provide a scheme for a quantum version of a penny flip game, based on the Heisenberg chain model. We study a protocol for playing a quantum game where two players implement their moves using control parameters of the model. We show that for a two-qubit spin chain it is possible to harness the complex structure of the system to obtain a higher mean payoff.

This paper is organized as follows. In Section 2 we review a classical penny flip (or bit flip) game including a quantum versus classical player scenario. In Section 3 we discuss the quantized version of the penny flip game played using control parameters of a single qubit. In Section 4 we extend the previous model of the penny flip game and discuss quantum strategies which can be achieved using an auxiliary qubit. In Section 5 we provide the summary of the presented work and give some concluding remarks.

2 Penny flip game

Penny flip game, also known as a bit flip game, provides one of the simplest examples used in many textbooks on game theory. It can be also easily used to show that by using quantum strategies, represented by unitary gates, one of the players can always win.

2.1 Classical version

Let us consider two parties (players) – Alice and Bob – playing a strategic game consisting of flipping a coin. At the beginning of the game the coin is faced heads up. The game consists of three moves. Alice plays first, then Bob and then Alice again. Each of them can chose to flip or not to flip the coin. The coin is hidden, so they cannot see it. Each of them does not know the actions of the other. The goal of Alice is that the coin ends facing tails up. The goal of Bob is that it ends heads up.

The payoff matrix for the penny-flip game is presented in Table 1, where N is used to denote the not-flipping strategy and F denotes the flipping strategy. One means that Alice has won, minus one means that Bob has won and the game is a zero-sum game.

The above scenario can be described mathematically in a language of quantum mechanics by using state vectors and unitary gates. In such description a coin is replaced by a qubit and we map a coin facing heads up to state $|0\rangle$, and facing tails up to $|1\rangle$. The flip operation corresponds to unitary matrix

	NN	NF	FN	FF
N	-1	1	1	-1
F	1	-1	-1	1

Table 1 Payoff matrix for the penny flip game.

$F = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ *i.e.* quantum *Not* gate, while not-flipping operator is described by the identity matrix, $N = \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The probability of winning for Alice and Bob can be calculated as the expectation value of the σ_z operator

$$\langle \sigma_z \rangle_{|c\rangle} = \text{tr}(|c\rangle\langle c| \sigma_z), \quad (1)$$

where $|c\rangle$ is the state of the coin after the second move of Alice.

One can easily see that in this scenario there is no winning strategy neither for Alice nor for Bob.

2.2 Quantum versus classical player

In order to show how quantum mechanics can be used to cheat in the penny-flip game one should note that in the above description Alice and Bob have only a very limited set of operations at their disposal.

Let us now use a qubit instead of a penny and allow one of the players, namely Alice, to use any unitary operation during her moves. In this situation any normalised linear combination of the base states, $\alpha|0\rangle + \beta|1\rangle$, $\alpha, \beta \in \mathbb{C}$ with $|\alpha|^2 + |\beta|^2 = 1$, describes an allowed state of a qubit representing a coin. Let us also assume that only Alice knows that the game is played with a qubit.

The above condition allows Alice to win with probability 1. She can rotate the qubit in any direction and prepare the state which is invariant with respect to any move Bob can use. To achieve this, she may chose to apply a Hadamard gate, $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, during her both moves. Bob still has only two possibilities: to flip (σ_x) or not to flip ($\mathbb{1}$). In these two cases we have

1. $H\sigma_x H|0\rangle \rightarrow |0\rangle$,
2. $H\mathbb{1}H|0\rangle \rightarrow |0\rangle$.

In the last step Alice can prepare a coin in state $|1\rangle$ by using σ_x operator. From the above it follows that Bob's actions have no influence whatsoever on the final state of the coin and thus the outcome of the game.

One should note that we have restricted ourselves to pure strategies. More detailed analysis of this game including mixed (probabilistic) strategies may be found in [17].

3 Qubit flip game

In the scheme described in Section 2.2 the player using quantum strategies was able to always win. This situation was possible since the player using

classical strategies was not able to explore the possible space of states. As such, it is natural to ask how the situation changes if both players are able to use quantum moves.

Let us assume that both Alice and Bob are now aware of the fact that they are playing the game using a qubit. Therefore now we have the following game

$$\Gamma = (\mathcal{H}, |\psi\rangle, S_A, S_B, P_A, P_B), \quad (2)$$

where $\mathcal{H} = \mathbb{C}^2$ is a Hilbert space, $|\psi\rangle = |0\rangle$ is an initial state, $S_A = U(2) \times U(2)$ the set of Alice's strategies, $S_B = U(2)$ the set of Bob's strategies and P_A, P_B are respectively Alice's and Bob's payoffs which are defined below.

The course of the game is as follows. Alice plays first – she applies her first element of strategy, U_{A_1} , on the initial state, then Bob plays applying his strategy U_B , finally Alice plays by applying the second element of her strategy, U_{A_2} . Then the projective measurement $\{O_{\text{heads}} \rightarrow |0\rangle\langle 0|, O_{\text{tails}} \rightarrow |1\rangle\langle 1|\}$ is performed. The final state of the game (before measurement) is

$$|\psi_f\rangle = U_{A_2} U_B U_{A_1} |0\rangle. \quad (3)$$

Alice's and Bob's payoff functions depend on the probability of measuring O_{tails} and O_{heads} respectively $P_A(S_A, S_B) = |\langle 1 | \psi_f \rangle|^2$, $P_B(S_A, S_B) = |\langle 0 | \psi_f \rangle|^2 = 1 - P_A(S_A, S_B)$.

3.1 Two quantum players with one qubit

Pure strategies in this game cannot be in Nash equilibrium, that is why in searching for optimality the players will use mixed strategies. To obtain Nash equilibrium the first or the second player can use the variety of mixed strategies. Here we give examples of two such strategies, namely *Pauli strategy* and *Haar strategy*. We will show that these strategies give Nash equilibrium.

- *Pauli strategy* is a mixed strategy, where a player chooses one of four unitary matrices $\{\mathbb{1}, i\sigma_x, i\sigma_y, i\sigma_z\}$ with equal probability.
- In *Haar strategy* a player chooses a random special unitary matrix distributed with natural probability measure invariant with respect to unitary transformation, *i.e.* the Haar measure.

We will show that if the second player uses one of the above strategies, then, no matter what action the first player will take, the probabilities of the success for both players are equal. We start with a definition.

Definition 1 A finite subset X of $U(d)$ is unitary t -design [20] if

$$\frac{1}{|X|} \sum_{U \in X} U^{\otimes t} \otimes (U^*)^{\otimes t} = \int_{U(d)} U^{\otimes t} \otimes (U^*)^{\otimes t} d\mu(U), \quad (4)$$

where μ is the Haar measure on a group $U(d)$.

Lemma 1 *Let X be a unitary 1-design. Then a mixed strategy such that the second player chooses to play a random element $U \in X$ chosen with equal probability, gives Nash equilibrium.*

Proof Let us denote the unitary matrices played by the first player as

$$U_{A_1} = \{a_{ij}\}_{i,j=1}^2, \quad U_{A_2} = \{b_{ij}\}_{i,j=1}^2. \quad (5)$$

Then the probability of success for the second player is given by

$$\bar{P}_B = \int_{U(2)} |\langle 0|U_{A_2}UU_{A_1}|0\rangle|^2 d\mu_X(U), \quad (6)$$

where μ_X is a uniform probability measure supported on X . Simple calculations give us

$$\bar{P}_B = \int_{U(2)} |\langle 0|U_{A_2}UU_{A_1}|0\rangle|^2 d\mu_X(U) \quad (7)$$

$$= \sum_{i,j=1}^2 |a_{1i}b_{j1}|^2 \int_{U(2)} |U_{ij}|^2 d\mu_X(U) \quad (8)$$

$$+ \sum_{\substack{i \neq k \\ j \neq l}} a_{1i}b_{j1}\overline{a_{ki}b_{l1}} \int_{U(2)} U_{ij}\overline{U_{kl}} d\mu_X(U). \quad (9)$$

Since X is a unitary 1-design, the values of the above integrals are the same as the value of integrals with respect to Haar measure [2]

$$\int_{U(2)} U_{ij}\overline{U_{kl}} d\mu(U) = \frac{1}{2}\delta_{ik}\delta_{jl}, \quad (10)$$

where μ is Haar measure on a group $U(2)$. Finally, since U_{A_1}, U_{A_2} are unitary, we get

$$\bar{P}_B = \frac{1}{2} \sum_{i,j=1}^2 |a_{1i}b_{j1}|^2 = \frac{1}{2}. \quad (11)$$

Thus, if the second player chooses to play a strategy given by a 1-design, no player can benefit by altering their strategy. \square

Since the Pauli strategy forms a unitary 1-design, one can see that the Pauli strategy gives Nash equilibrium.

The same holds true for the Haar strategy, but in this case to obtain the result one needs to notice that

$$\int_{U(2)} U \otimes U^* d\mu_S(U) = \int_{U(2)} U \otimes U^* d\mu(U), \quad (12)$$

where μ_S is a Haar measure on a group $SU(2)$.

From the above considerations one can see that to achieve Nash equilibrium, Bob can take any mixed strategy, in which he chooses unitary matrices with the distribution ν satisfying the condition

$$\int_{U(2)} U \otimes U^* d\nu(U) = \int_{U(2)} U \otimes U^* d\mu(U). \quad (13)$$

Moreover, it is easy to see the following.

Proposition 1 *If the second player uses a mixed strategy by choosing uniformly from a finite set X , then this strategy gives a Nash equilibrium if and only if set X is a unitary 1-design.*

The above result suggests that, if Bob aims to play optimally, he should always play a strategy given by a unitary 1-design. In the following we assume that he always uses the Pauli strategy.

3.2 Hamiltonian-based implementation on one qubit

The rules of the qubit-flip game can be described to reflect the constraints that are imposed upon the physical realizations of the game. The players will be allowed to program a quantum device that represents a one-qubit system. They will be constrained by the execution time of the program and the set of available instructions.

Let us assume that, instead of being able to execute a fixed unitary transformation, Alice and Bob are now allowed to play the bit-flip game by manipulating the control parameters in the Hamiltonian $H = H_c(t)$. Note that since there is no drift term in this equation, we assume that players have complete control over the system. They are only constrained by maximal fixed time T of the game turn. During this time Alice and Bob can change values of the parameters h_z and h_y .

The realisation of the game in this set-up is as follows. Alice moves as the first player and applies the series of control to the system, then Bob performs his sequence of controls and, finally, Alice performs her second sequence as the last move in the game. Each move is described by a sequence of parameters $h_y(0), h_z(\Delta t), h_y(2\Delta t), \dots, h_i((N-1)\Delta t)$, with $h_i = h_y$ or $h_i = h_z$ depending on the length of the sequence. We assume that players cannot use h_z and h_y simultaneously. We will only consider the case where the time for which the control is applied is equal for all the controls.

The final state of the game can be obtained from Eq. (3) with

$$U_{A_1} = e^{-ih_i^{A_1}((N_{A_1}-1)\Delta t)\sigma_i} \dots e^{-ih_z^{A_1}(\Delta t)\sigma_z} e^{-ih_y^{A_1}(0)\sigma_y}, \quad (14)$$

$$U_B = e^{-ih_i^B((N_B-1)\Delta t)\sigma_i} \dots e^{-ih_z^B(\Delta t)\sigma_z} e^{-ih_y^B(0)\sigma_y}, \quad (15)$$

$$U_{A_2} = e^{-ih_i^{A_2}((N_{A_2}-1)\Delta t)\sigma_i} \dots e^{-ih_z^{A_2}(\Delta t)\sigma_z} e^{-ih_y^{A_2}(0)\sigma_y}. \quad (16)$$

	ξ_1	ξ_2	ξ_3
1	0	0	0
$i\sigma_x$	$\pi/4$	$-\pi/2$	$\pi/4$
$i\sigma_y$	0	$-\pi/2$	0
$i\sigma_z$	$-\pi/4$	0	$-\pi/4$

Table 2 Control parameters for realizing the Pauli strategy [3].

To obtain the Pauli or Haar strategy the second player may calculate control sequences using Euler decomposition of $SU(2)$. For a given special unitary matrix one can write

$$\begin{pmatrix} e^{i\phi} \cos(\theta) & e^{i\psi} \sin(\theta) \\ -e^{-i\psi} \sin(\theta) & e^{-i\phi} \cos(\theta) \end{pmatrix} = e^{i\frac{\phi+\psi}{2}\sigma_z} e^{i\theta\sigma_y} e^{i\frac{\phi-\psi}{2}\sigma_z}. \quad (17)$$

Thus to obtain the Pauli strategy the second player chooses uniformly one of four matrices from the Pauli strategy and takes control parameters (ξ_1, ξ_2, ξ_3) from Table 2.

The player applies parameters as a control sequence with constant factor $\frac{3}{T}$, each of them for one-third of total time T , obtaining

$$e^{-i\xi_3\sigma_z} e^{-i\xi_2\sigma_y} e^{-i\xi_1\sigma_z}, \quad (18)$$

which is one of four matrices in the Pauli strategy.

On the other hand, to obtain the Haar strategy the second player may choose parameters ξ_1, ξ_2, ξ_3 as

$$\xi_1 = -\frac{\phi - \psi}{2}, \quad \xi_2 = -\arcsin(\sqrt{p}), \quad \xi_3 = -\frac{\phi + \psi}{2}, \quad (19)$$

where ϕ, ψ, p are independent random variables with $\phi, \psi \sim U(0, 2\pi)$ and $p \sim U(0, 1)$. The player applies the chosen parameters with constant factor $\frac{3}{T}$, each of them for one-third of total time T , obtaining a random special unitary matrix with Haar distribution [21] *i.e.* matrix from the Haar strategy.

4 Cheating with auxiliary spins

Let us now consider an extension of the qubit-flip scenario where one of the players, namely Alice, is aware of the fact that the system used to play the game is composed of two qubits. In this situation one can ask if, analogously to the situation in classical versus quantum player, she can use this knowledge to get a better mean payoff.

4.1 Description of the scheme

In the following we are interested in a specific class of systems where the Hamiltonian operator of the system is given as

$$H(t) = H_0 + H_c(t) \quad (20)$$

where H_0 is the drift Hamiltonian and H_c is the control Hamiltonian.

In the case of time-independent Hamiltonian ($H_c(t) \equiv 0$) the unitary matrix U describing the time evolution of the system in question is obtained from the Hamiltonian operator of the system by $U = \exp(-itH_0)$ for a given t .

However, we are interested in the situation where $H_c(t) \neq 0$ and we are going to consider the control Hamiltonian, such that control parameters are piecewise constant in time. In this situation the resulting unitary evolution is of the form

$$U = \prod_{n=0}^{N-1} e^{-i\Delta t(H_0 + H_c(n\Delta t))} = U((N-1)\Delta t) \dots U(\Delta t)U(0). \quad (21)$$

The Hamiltonian operator of the system used to play the game in such case is given as in Eq. (20) with

$$H_0 = J(\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y + \sigma_z \otimes \sigma_z) \quad (22)$$

and

$$H_c(t) = h_y(t)\sigma_y \otimes \mathbb{1} + h_z(t)\sigma_z \otimes \mathbb{1}. \quad (23)$$

Here $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the parameter $J \in \mathbb{R}$ is the coupling constant [11].

4.2 Mimicking one-spin behaviour

If Alice aims to cheat against Bob by using auxiliary spins, the first challenge she has to face is to convince Bob that the system is constructed in such a way that both players can win with equal probability. Let us assume that Bob can study the system used to play the game with a series of experiments before actually playing the game. The only possible experiment Bob can make is to play the game against Alice with the best possible strategy and observe his mean payoff. If the payoff is equal to $1/2$, then he must assume that both players have an equal probability of winning.

In order to achieve this effect, Alice must choose the value of J/T in such way, that if both players use the Pauli strategy they cannot distinguish if they are using a device composed of one or two qubits using the information about the mean payoff only.

Let us assume that $T = 1$. If the system (the spin chain) used to play the game is composed of two qubits only, Alice is able to convince Bob that they are using one qubit only.

In Fig. 1 Bob's mean payoffs for different values of the coupling constant J are presented. In the case of a spin chain composed of two spins, the smallest value of J greater than zero, such that the condition for the game appears to be fair, is equal to $J/T \approx 4.10469$.

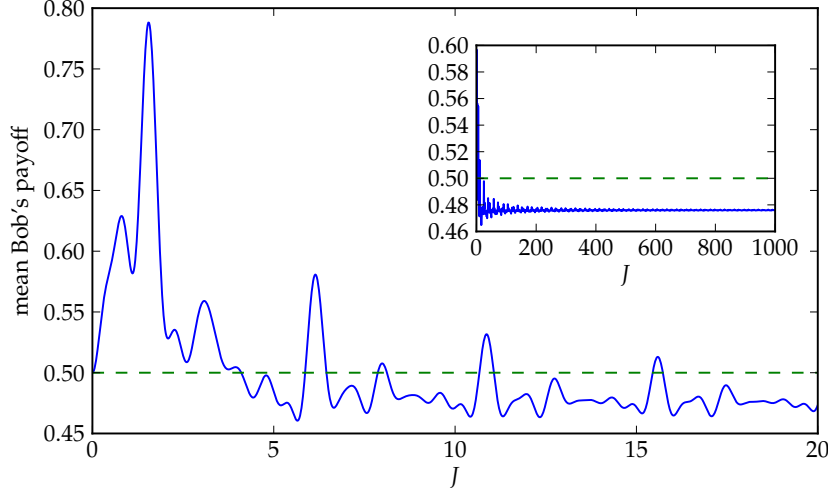


Fig. 1 Mean value of Bob's payoff in function of spin chain coupling strength J when both players use Pauli strategy. The inset plot shows the behaviour of the payoff for very large values of J .

4.3 Mean payoff for Alice

Having the value of J/T fixed, we can assume that Bob always uses the optimal strategy, *i.e.* he aims to achieve the probability of winning equal to $1/2$. In the following we assume that he always plays the Pauli strategy and applies three controls.

Let be given H_0 , the drift Hamiltonian, which is known to Alice only and T , the time for each move, known to both players. Alice's goal is to optimize the following functional

$$P_A(\mathbf{h}_1, \mathbf{h}_2) = \frac{1}{4} \sum_{i=1}^4 \langle 1 | \text{tr}_2(|\psi_i^f\rangle\langle\psi_i^f|) | 1 \rangle \quad (24)$$

with respect to control vectors \mathbf{h}_1 and \mathbf{h}_2 . The final state of the game $|\psi_i^f\rangle$ reads

$$|\psi_i^f\rangle = U(\mathbf{h}_2)V_iU(\mathbf{h}_1)|00\rangle, \quad (25)$$

where matrices $U(\mathbf{h}_1)$ and $U(\mathbf{h}_2)$ are defined as in Eq. (21). Matrices V_i describe one of Bob's possible moves,

$$V_i = e^{-i\frac{T}{3}(H_0 + \frac{3}{T}\xi_3^{(i)}\sigma_z \otimes \mathbf{1})} e^{-i\frac{T}{3}(H_0 + \frac{3}{T}\xi_2^{(i)}\sigma_y \otimes \mathbf{1})} e^{-i\frac{T}{3}(H_0 + \frac{3}{T}\xi_1^{(i)}\sigma_z \otimes \mathbf{1})}, \quad (26)$$

where $\xi_k^{(i)}$, $k = 1, 2, 3$ are control parameters from i -th row of Table 2.

4.3.1 Optimal strategy for Alice

In order to find optimal control sequences allowing to maximize the final output we have used nonlinear conjugate gradient algorithm [15] implemented as a part of SciPy open-source mathematical software distributed with modified BSD license [12].

In Fig. 2 the sequences of control pulses allowing Alice to obtain mean payoff equal approximately to 1 are presented. It is easy to observe that she is able to exploit the internal structure of the system by using a very short sequence of control pulses. Additionally, the gain obtained by increasing the length of control sequences is very small.

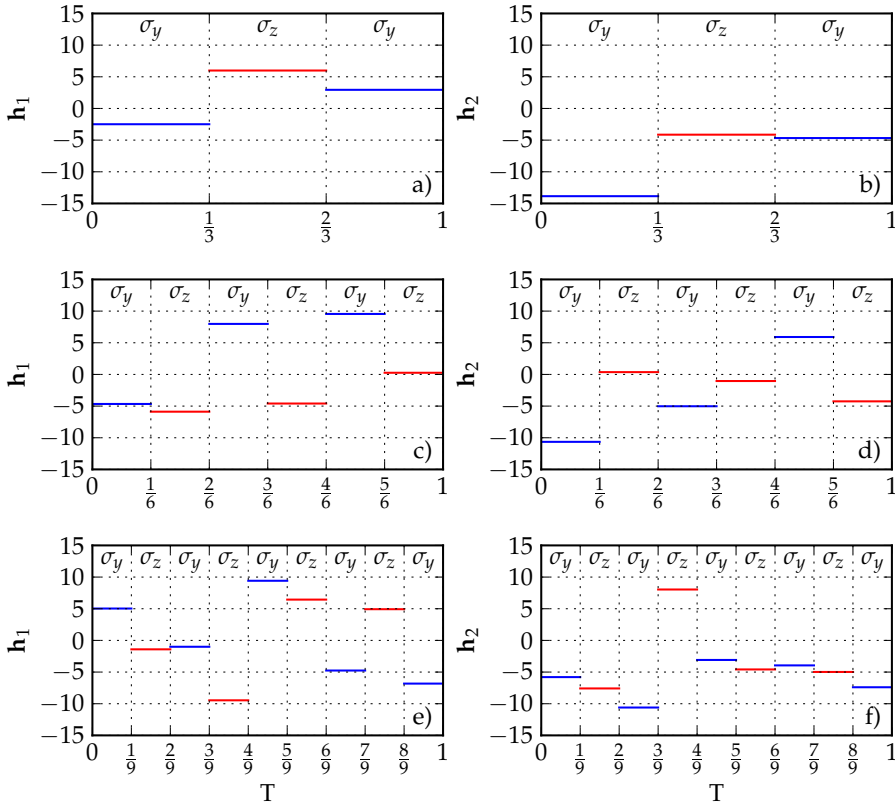


Fig. 2 Controls that maximize Alice's payoff. Panels a) and b) show the sequence of pulses that generate gates U_{A_1} and U_{A_2} respectively when Alice chooses to apply only three controls to run her strategy. In this case her probability of winning is equal to 0.97. Similarly panels c) and d) represent her strategy using 6 pulses per gate; in this case her probability of winning is 0.988. Panels e) and f) represent 9-pulse strategy which gives the probability of winning 0.999.

From the Fig. 2 one can conclude that the knowledge of the dimensionality of the system can be easily used to cheat against the opponent. This situation resembles the one described in Sec. 2.2, where one of the players was playing using the classical moves only. In our case, however, both players are able to exploit quantum strategies.

4.3.2 Optimal strategy for Bob

Let us now assume that the device used to play the game was prepared not by Alice, but by Bob. In such situation only Bob is aware of the internal structure of the system and he may use it to increase his payoff.

We can distinguish the following two scenarios where Bob aims to cheat against Alice:

1. Bob uses one auxiliary qubit and he tries to find the best possible sequence of controls;
2. Bob adds more auxiliary qubits and tries to find the best sequence of control pulses.

Optimization over the number of controls We may assume that now Alice is using the Pauli strategy. The J/T was chosen so that both players have no reasons for suspecting that the system has higher dimensionality.

In Fig. 3 the sequences of control pulses allowing Bob to achieve mean payoff greater than $1/2$ are presented.

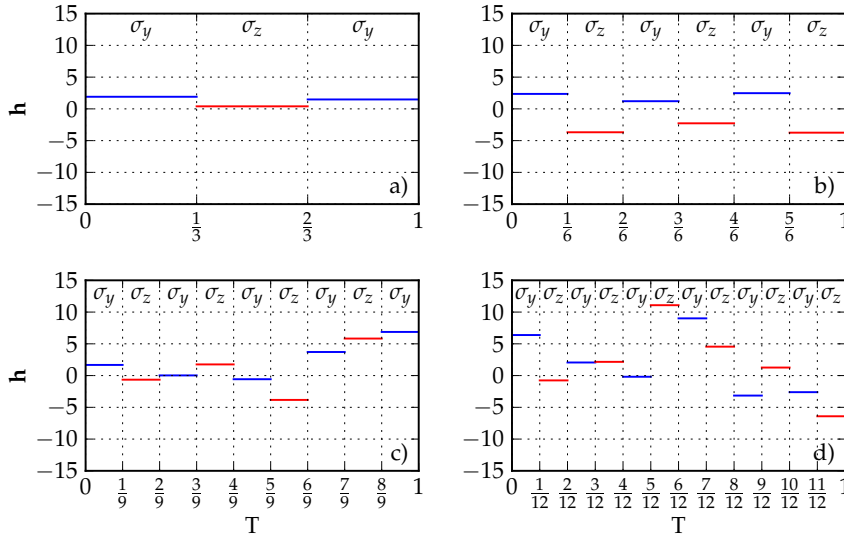


Fig. 3 Controls that maximize Bob's payoff. Panel a) shows the control vector that generates Bob's strategy when using 3 control parameters, in this case his payoff is 0.713. Panels b), c) and d) show Bob's controls of length 6, 9 and 12 giving payoffs 0.804, 0.8241 and 0.8246 respectively.

Optimization with additional qubits Let us now consider the scenario where Bob is able to add more auxiliary qubits and build a longer spin-chain. Unfortunately, in this situation it is impossible to find J/T such that the mean payoff for both players using the Pauli strategy is $1/2$. For this reason we must assume that Bob convinced Alice to play the qubit flip game using his device without allowing her to check it by playing with the Pauli strategy.

Chain length	2	3	4	5	6	7
Maximal payoff	0.713	0.920	0.925	0.901	0.901	0.951

Table 3 Bob's maximal payoff for a different size of a chain in the case of using three control pulses.

The maximal payoffs obtained by Bob in this scenario are presented in Table 3.

5 Concluding remarks

We have discussed some possible realizations of a penny flip game in a quantum system. First, we have assumed that both players are aware of the fact that they play the game using a qubit. In the next step we have extended the model by adding an additional qubit.

In the second case only the first player was aware of the fact that the game is played on a two-qubit system and of the type of interaction between qubits. We have shown that in such a situation she is able to achieve a mean payoff equal to almost 1. Moreover, it is enough to use one auxiliary qubit (spin) only to achieve this effect. This shows that in the situation when one of the parties is not aware of the dimensionality of the system, the other can easily use this fact to play unfairly.

To show that also the second player can gain by harnessing the dimensionality of the system we have introduced a scenario where he tries to cheat against the first player. We have shown that the payoff of the second player grows with the number of controls and with the number of auxiliary qubits.

The presented work leaves open a problem of the behaviour of the game in the situation where players, or at least one of them, are able to use more general strategies, namely strategies described by quantum channels. This would also allow to incorporate the influence of noise on the course of the proposed game.

Acknowledgements Work by P. Gawron was supported by the Polish Ministry of Science and Higher Education under the grant number IP2010 009770, J.A. Miszczak was supported by Polish National Science Centre under the project number N N516 475440, Z. Puchała was supported by the Polish National Science Centre under the research project N N514 513340.

References

1. Chandrashekar, C., Banerjee, S.: Parrondo's game using a discrete-time quantum walk. *Phys. Lett. A* **375**(14), 1553 – 1558 (2011). DOI 10.1016/j.physleta.2011.02.071
2. Collins, B., Śniady, P.: Integration with respect to the haar measure on unitary, orthogonal and symplectic group. *Commun. Math. Phys.* **264**, 773–795 (2006). DOI 10.1007/s00220-006-1554-3
3. D'Alessandro, D.: Introduction to quantum control and dynamics. Chapman & Hall (2008)
4. Eisert, J., Wilkens, M., Lewenstein, M.: Quantum games and quantum strategies. *Phys. Rev. Lett.* **83**(15), 3077–3080 (1999). DOI 10.1103/PhysRevLett.83.3077
5. Flitney, A.P., Abbott, D.: Quantum version of the Monty Hall problem. *Phys. Rev. A* **65**(6), 062,318 (2002). DOI 10.1103/PhysRevA.65.062318
6. Flitney, A.P., Abbott, D.: Quantum games with decoherence. *J. Phys. A: Math. Gen.* **38**(2), 449–459 (2005). DOI 10.1088/0305-4470/38/2/011
7. Gawron, P.: Noisy quantum Monty Hall game. *Fluct. Noise Lett.* **09**(01) (2010). DOI 10.1142/S0219477510000034
8. Gawron, P., Klamka, J., Winiarczyk, R.: Noise effects in quantum search algorithm from the computational complexity point of view. *Int. J. Appl. Math. Comput. Sci.* (2011). In press
9. Gawron, P., Miszczak, J.: Quantum implementation of Parrondo paradox. *Fluct. Noise Lett.* **5**, L471–L478 (2005). DOI 10.1142/S0219477505002902
10. Gawron, P., Miszczak, J., Ślaskowski, J.: Noise effects in quantum magic squares game. *Int. J. Quantum Inf.* **06**(6 Supp 1.) (2008). DOI 10.1142/S0219749908003931
11. Heule, R., Bruder, C., Burgarth, D., Stojanović, V.: Local quantum control of Heisenberg spin chains. *Phys. Rev. A* **82**(5), 052,333 (2010). DOI 10.1103/PhysRevA.82.052333
12. Jones, E., Oliphant, T., Peterson, P., et al.: SciPy: Open source scientific tools for Python (2001–). URL <http://www.scipy.org/>
13. Košík, J., Miszczak, J.A., Bužek, V.: Quantum Parrondo's game with random strategies. *J. Mod. Opt.* **54**, 2275–2287 (2007)
14. Meyer, D.A.: Quantum strategies. *Phys. Rev. Lett.* **82**(5), 1052–1055 (1999). DOI 10.1103/PhysRevLett.82.1052
15. Nocedal, J., Wright, S.: Numerical Optimization. Springer-Verlag, New York, U.S.A (1999)
16. Pakula, I.: Quantum gambling using mesoscopic ring qubits. *Phys. Stat. Sol. B* **244**(7), 2513–2515 (2007). DOI 10.1002/pssb.200674643
17. Piotrowski, E.W., Ślaskowski, J.: An invitation to quantum game theory. *Int. J. Theor. Phys.* **42**, 1089 – 1099 (2003). DOI 10.1023/A:1025443111388
18. Ramzan, M., Khan, M.: Distinguishing quantum channels via magic squares game. *Quantum Inf. Process.* **9**(6), 667–679 (2010). DOI 10.1007/s11128-009-0155-4
19. Ramzan, M., Khan, M.: Decoherence and entanglement degradation of a qubit-qutrit system in non-inertial frames. *Quantum Inf. Process.* pp. 1–12 (2011). DOI 10.1007/s11128-011-0257-7
20. Roy, A., Scott, A.: Unitary designs and codes. *Designs Codes Cryptogr.* **53**(1), 13–31 (2009). DOI 10.1007/s10623-009-9290-2
21. Życzkowski, K., Kuś, M.: Random unitary matrices. *J. Phys. A: Math. Gen.* **27**(12), 4235 (1994). DOI 10.1088/0305-4470/27/12/028